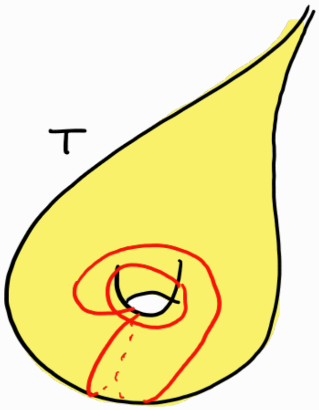
A proof of McShane's identity via Markoff triples

The aim of this lecture is to give a combinatorial proof of McShane's identity, due to Bowditch.

Let T be a hyperbolic once-punctured torus and

$$C = \{\text{simple closed geodesics on } T\}_{/\text{orientation}}.$$

We shall prove the following.



Theorem: (McShane identity)

$$\sum_{\gamma \in C} \frac{1}{1+e^{\ell(\gamma)}} = \frac{1}{2}.$$

I) Two actions of $PSL(2;\mathbb{Z})$

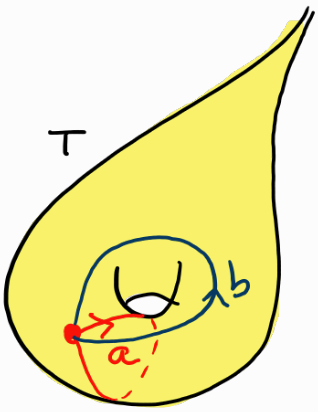
$$GL(2;\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ a,b,c,d \in \mathbb{Z},\ ad-bc = \pm 1 \right\}$$

$$SL(2;\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ a,b,c,d \in \mathbb{Z},\ ad-bc = 1 \right\}$$

$$PSL(2;\mathbb{Z}) = SL(2;\mathbb{Z})/_{\pm I_2}$$

1) Action on simple closed curves

Let T be a once-holed torus. We fix generators (a,b) of $\pi_1(T) = \mathbb{F}_2 = \langle a, b \rangle$.



We recall that the extended mapping class group of T is

$$\mathrm{Mod}^{\pm}(T) := \{\text{homeo } T \circlearrowleft\}_{/\text{homotopy}}$$

and $\mathrm{Mod}(T)$ is the sub-group of orientation-preserving homeo.

Mod(T) naturally acts on free homotopy classes of closed curves by letting $\phi([\gamma]) = [\phi(\gamma)]$ for $\phi \in \mathrm{Mod}(T)$.

We recall two results following easily from the previous lecture.

Lemma: $PSL(2;\mathbb{Z})$ acts transitively on

$$C = \{\text{simple closed curves on } T\} / {\substack{\text{free homotopy}\\ \text{orientation}}},$$

and the stabilizer of a is $\simeq \left\langle \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$.

Proof: We saw last time that $\mathrm{Mod}^{\pm}(T) \simeq GL(2;\mathbb{Z})$ acts transitively on $\{\text{simple closed curves on } T\} / \text{free homotopy}$.

The identification $\mathrm{Mod}^{\pm}(T) \simeq GL(2;\mathbb{Z})$ comes from writing the matrix M_ϕ of $\phi \in \mathrm{Mod}^{\pm}(T)$ acting on $H_1(T;\mathbb{Z}) \simeq \mathbb{Z}^2$. Then ϕ preserves orientation $\Leftrightarrow$ $\det M_\phi = 1$ hence $\mathrm{Mod}(T) \approx SL(2,\mathbb{Z})$.

The first part of the claim then follows by quotient by orientations.

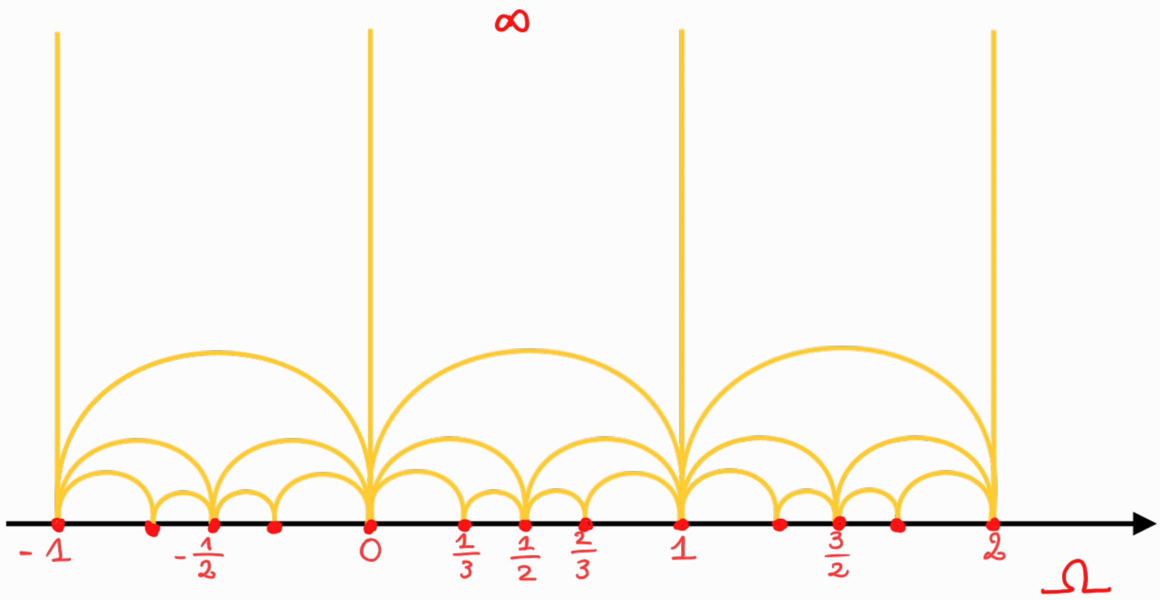
The generators a, b of $\pi_1(T)$ give us a basis for homology, and then the stabilizer of a is

$$\left\{ M \in PSL(2;\mathbb{Z}) : M\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} = \left\{ \pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$$ as we claimed.

② Action on the Farey triangulation

The group $PSL(2;\mathbb{Z}) \subset PSL(2;\mathbb{R})$ acts on $\mathbb{H}$ by isometries.

Starting from the ideal triangle of vertices $0, 1, \infty$ and shifting by the elements of $PSL(2;\mathbb{Z})$, we obtain the Farey triangulation:



Let Ω denote the set of vertices of the Farey triangulation.

<u>Lemma</u>: $PSL(2;\mathbb{Z})$ acts on Ω transitively and the stabilizer of $\infty \in \Omega$ is $\left\{ \pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$.

<u>Proof</u>: This follows from the action of $PSL(2;\mathbb{Z})$ on the triangulation, and the fact that any $\phi \in PSL(2;\mathbb{Z})$ sending ∞ to ∞ is in $\left\{ \pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$.

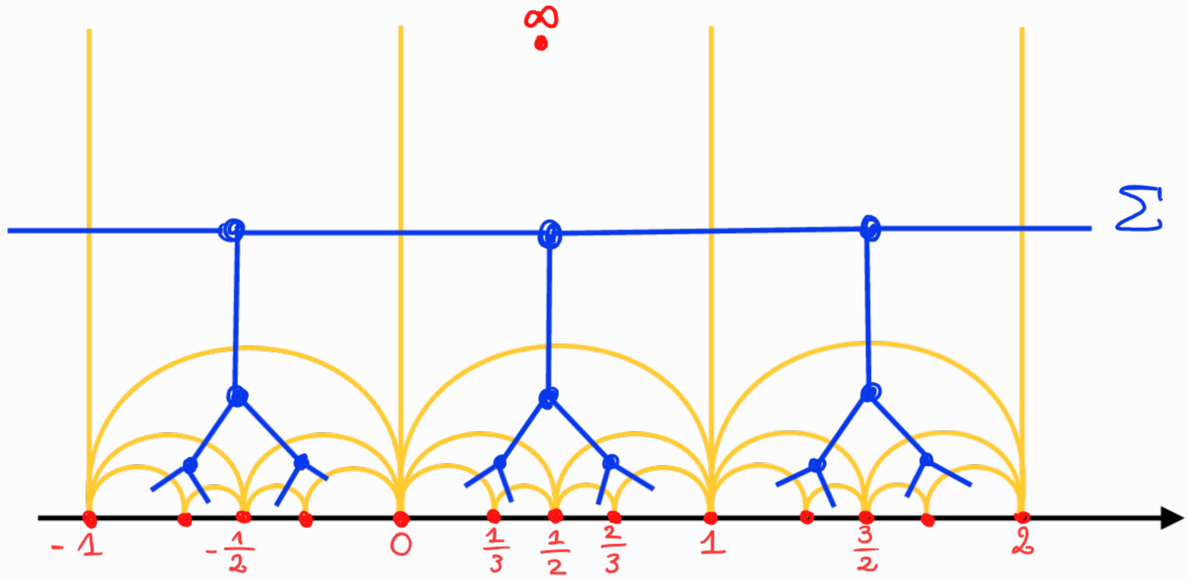
As a consequence, there is a canonical bijection $\begin{array}{ccc} \Omega & \longrightarrow & C \\ x & \longmapsto & \gamma_x \end{array}$ which identifies ∞ and a.

I) The Farey tree and Markoff maps

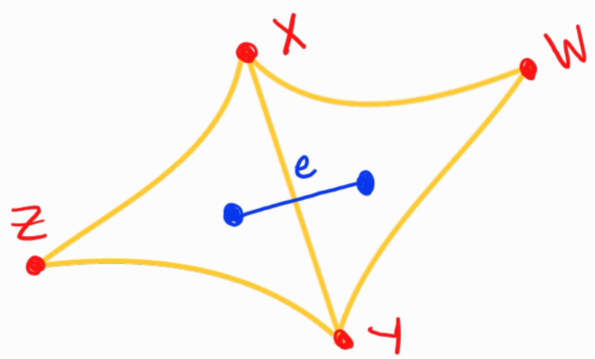
We define the Farey tree Σ to be the dual of the Farey triangulation, i.e. $\Sigma = (V(\Sigma), E(\Sigma))$ is a graph embedded in $\mathbb{H}$ with

- vertex set $V(\Sigma) = \{$ triangles of triangulation $\}$
- edge set $E(\Sigma) = \{$ sides of triangulation $\}$.

This is an infinite 3-regular tree embedded in $\mathbb{H}$.



Let $e \in E(\Sigma)$. The edge e corresponds to two adjacent ideal Farey triangles. Those two triangles share a pair of vertices $\{X, Y\} \in \Omega$. We denote by $\{Z, W\}$ the remaining two vertices.



We make the following key definition.

Definition: A map $f : \Omega \to (2,\infty)$ is called a Markoff map if, for any e meeting $\{X,Y\}$ and $\{Z,W\}$, if $(x,y,z,w) = f(X,Y,Z,W)$,

(i) $x^2 + y^2 + z^2 = xyz$

(ii) $xy = z + w$.

Note the similarity of (i) with the Markoff equation.

McShane's identity will be deduced from the bijective map $\begin{array}{l}\Omega \to C \\ X \mapsto \gamma_X\end{array}$ using a combination of the following two results.

Lemma: Let T be a hyperbolic once-holed torus. Then,

$$f_T : \begin{array}{l}\Omega \longrightarrow (2,\infty) \\ X \longmapsto 2\cosh\left(\frac{\ell_T(\gamma_X)}{2}\right)\end{array}$$

is a Markoff map.

Theorem: If $f : \Omega \to (2,\infty)$ is a Markoff map, then

$$\sum_{X\in\Omega} h(f(X)) = \frac{1}{2} \quad \text{where } h(x) := \frac{1-\sqrt{1-4/x^2}}{2}$$

Indeed, for any ℓ,

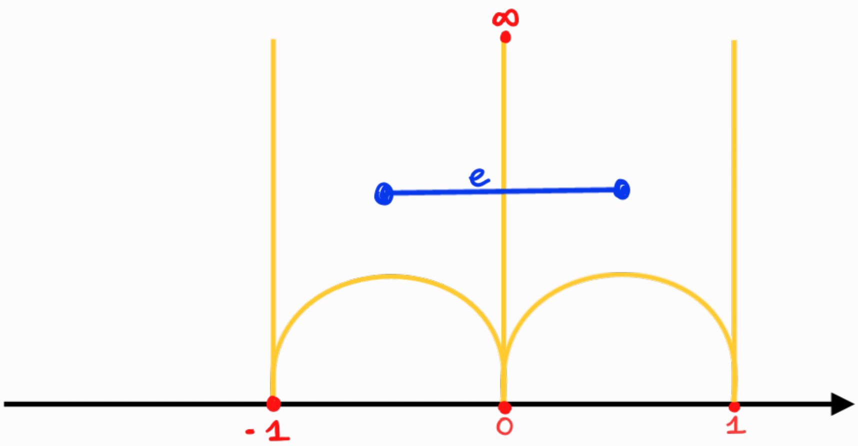
$$h\left(2\cosh\frac{\ell}{2}\right) = \frac{1}{2}\left(1 - \sqrt{1 - \frac{1}{\cosh^2 \ell/2}}\right) = \frac{1}{2}\left(1 - \frac{\sinh(\ell/2)}{\cosh(\ell/2)}\right)$$

$$= \frac{e^{-\ell/2}}{e^{\ell/2} + e^{-\ell/2}} = \frac{1}{1+e^{\ell}}.$$

III Proof of the lemma

Let $e \in E(\Sigma)$ meeting $\{X,Y\}$ and $\{Z,W\}$.

WLOG we assume $X=\infty$, $Y=0$, $Z=1$ and $W=-1$, by acting by an element of $PSL(2;\mathbb{Z})$.



We have the dialogue between the 3 actions :

	$\begin{pmatrix}1\\0\end{pmatrix} \in H_1(T,\mathbb{Z})$	$a \in C$	$\infty \in \Omega$
$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$	b	$\frac{0-1}{\infty+0} = 0$
$\begin{pmatrix}1 & -1\\1 & 0\end{pmatrix}$	$\begin{pmatrix}1\\1\end{pmatrix}$	ab	$\frac{\infty-1}{\infty+0} = 1$
$\begin{pmatrix}1 & 1\\-1 & 0\end{pmatrix}$	$\begin{pmatrix}1\\-1\end{pmatrix}$	ab^{-1}	$\frac{\infty+1}{-\infty+0} = -1$

which allows us to identify X,Y,Z,W with a, b, ab, ab^{-1}.

Recall that a,b correspond to matrices $A,B \in SL(2;\mathbb{R})$, which we pick to have positive trace. Then,

$$f_T(X) = 2\cosh\left(\frac{\ell(a)}{2}\right) = \operatorname{tr} A \quad \text{and} \quad f_T(Y) = 2\cosh\left(\frac{\ell(b)}{2}\right) = \operatorname{tr}(B).$$

Lemma: $\operatorname{tr}([A,B]) = -2$.

Proof: $[A,B] = ABA^{-1}B^{-1}$ corresponds to the closed curve around the cusp, so it is parabolic of trace ± 2.

WLOG $A = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$ and $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

so $\operatorname{tr}[A,B] = 2\alpha\delta - \left(a^2 + \frac{1}{a^2}\right)\beta\gamma = 2 - \left(a^2 + \frac{1}{a^2} + 1\right)\beta\gamma$ so $\beta\gamma = 0$

and hence A, B share a fixed point which is impossible

since $\mathbb{F}_2 = \langle A, B \rangle$. $\square$

We saw last lecture that in this case Cohn proved

$$(\operatorname{tr} A)^2 + (\operatorname{tr} B)^2 + (\operatorname{tr} AB)^2 = \operatorname{tr} A \cdot \operatorname{tr} B \cdot \operatorname{tr}(AB).$$

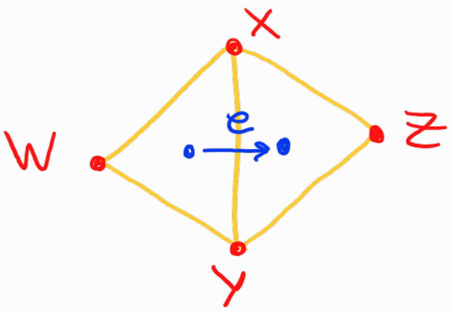
In particular, $\operatorname{tr}(AB) > 0$ so $f_T(z) = \operatorname{tr}(AB)$ and we have (i).

Similarly, $\operatorname{tr}(AB^{-1}) > 0$ so $f_T(W) = \operatorname{tr}(AB^{-1})$ and (ii) is obtained

from $\operatorname{tr} A \operatorname{tr} B = \operatorname{tr}(AB) + \operatorname{tr}(AB^{-1})$.

IV Proof of the theorem

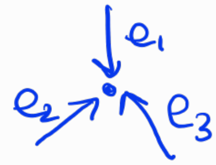
We denote as $\vec{E}(\Sigma)$ the set of directed edges of Σ. Each $\vec{e}$ has a head and a tail. We will denote z the vertex in the triangle corresponding to the head.



Notation: For $\vec{e} \in \vec{E}(\Sigma)$, define $F(\vec{e}) = \frac{z}{xy} = \frac{f(Z)}{f(X)f(Y)}$.

Then we can rewrite the relations (i) and (ii) as

(i') for any $v \in V(\Sigma)$, if $\vec{e_1}, \vec{e_2}, \vec{e_3}$ are the 3 oriented edges of head v, then $F(\vec{e_1}) + F(\vec{e_2}) + F(\vec{e_3}) = 1$



(ii') for $\vec{e} \in \vec{E}(\Sigma)$, if $-\vec{e}$ is the edge with opposite orientation, then $F(\vec{e}) + F(-\vec{e}) = 1$.

Definition: Let $\mathcal{G} \subset \Sigma$ be a finite subtree. We define

$$C(\mathcal{G}) = \{ \vec{e} \in \vec{E}(\Sigma) : \text{head } \vec{e} \in \mathcal{G},\ \text{tail } \vec{e} \notin \mathcal{G} \}$$

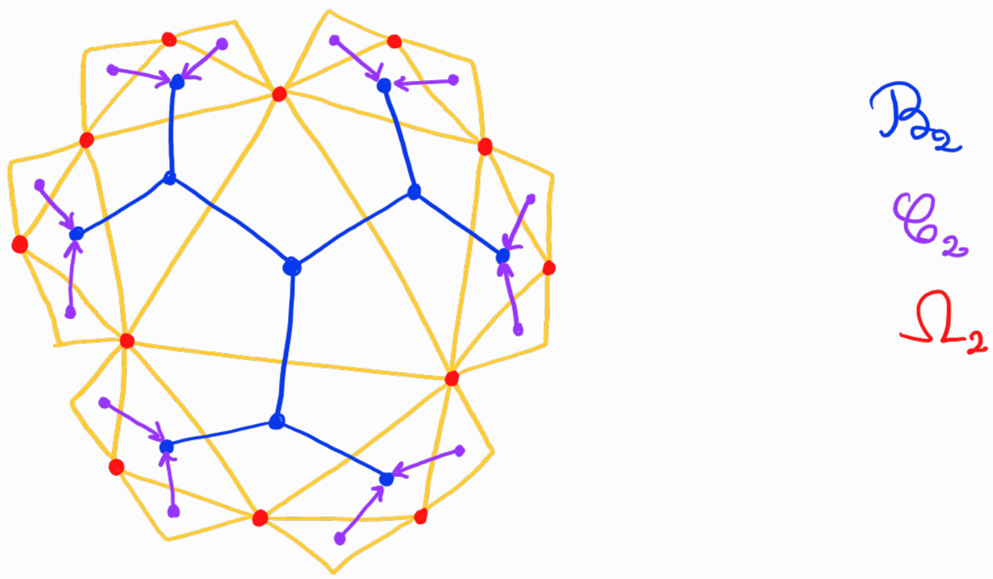
The conditions (i') and (ii') have the following rewriting.

Lemma: For any finite subtree $\mathcal{G} \subset \Sigma$, $\sum_{\vec{e} \in C(\mathcal{G})} F(\vec{e}) = 1$.

Notation: For $r \in V(\Sigma)$ and $n \geq 0$ we define

$$B_n(r) = \{ v \in V(\Sigma) : d(r, v) \leq n \} \qquad \mathcal{C}_n(r) = C(B_n(r))$$

and $\Omega_n(r) = \{ X \in \Omega : X \text{ is a vertex of an element of } B_n(r) \}$.



For $n=0$, $\mathcal{B}_0(r) = r$, $\mathcal{C}_0(r) = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ as before and $\Omega_0(r)$ is the set of vertices of the triangle r.

We are ready to prove the upper bound.

Lemma: $\displaystyle\sum_{X\in\Omega} h(f(X)) \leq \frac{1}{2}$.

Proof:

Let $\vec{e}\in\vec{E}$. We have by (i'), $z^2 - xyz + x^2+y^2 = 0$ so

$$z = \frac{xy \pm \sqrt{x^2y^2 - 4(x^2+y^2)}}{2}.$$

In particular,

$$F(\vec{e}) = \frac{z}{xy} \geq \frac{xy - \sqrt{x^2y^2-4(x^2+y^2)}}{2xy} = \frac{1-\sqrt{1-4\left(\frac{1}{x^2}+\frac{1}{y^2}\right)}}{2} =: h(x,y)$$

We check that $h(x,y) \geq h(x) + h(y)$.

Let $r\in V(\Sigma)$ and $n\geq 0$. Summing this for $\vec{e}\in\mathcal{C}_n(r)$ yields

$$1 = \sum_{\vec{e}\in\mathcal{C}_n(r)} F(\vec{e}) \geq \sum_{\vec{e}\in\mathcal{C}_n(r)} h(x)+h(y) = 2\sum_{X\in\Omega_n(r)} h(f(X))$$

which leads to the conclusion taking $n\to\infty$. $\square$

The lower bound requires more work.

Notation: We fix a subset $\vec{E}_+ \subset \vec{E}(\Sigma)$ of oriented edges, containing exactly one orientation of each $e\in E(\Sigma)$, such that for $\vec{e}\in\vec{E}_+$, $z\leq w$.

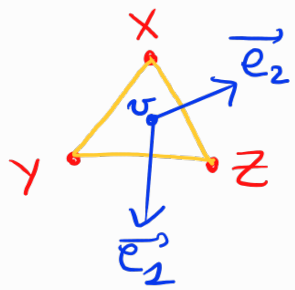
Exercise: $z\leq w \iff 2z\leq xy \iff xy\leq 2w$.

Lemma: There exists a unique vertex $r \in V(\Sigma)$ s.t. every edge $\vec{e} \in \vec{E}_+$ points towards r.

Here we will cheat a bit and assume $f = f_T$ for a once-holed torus T.

The argument can be made to work in general but the proof is longer.

Proof:



- Let $v \in V(\Sigma)$. Assume $\vec{e_1}$ and $\vec{e_2}$ leave v.

  Then $yz \leq 2x$ and $xz \leq 2y$

  so $z^2 \leq 4$ which is absurd since $z > 2$.

  Hence no vertex can be the tail of two $\vec{e_1}, \vec{e_2} \in \vec{E_+}$.

- Assume by contradiction that no vertex satisfies the claim.

  Using the first point, we can generate a sequence $(\vec{e_i})_{i \geq 0}$ of elements of $\vec{E}_+$ s.t. for all i, the head of $\vec{e_i}$ is the tail of $\vec{e}_{i+1}$.

  Then for $i \geq 0$, $z_i \leq w_i$. Since $z_i = w_{i+1}$, we obtain that $(w_i)_{i \geq 0}$ is a non-increasing sequence in $(2, \infty)$. As a consequence, there exists $L > 0$ and an infinite family of distinct simple closed geodesics on T s.t. $\ell_T(\gamma) \leq L$. This contradicts the fact that Γ is Fuchsian hence discrete. $\square$

We are now ready to conclude.

Proof: Take r given by the lemma. Then one vertex of the triangle r minimizes $f(X)$ globally, hence

$$\mu := \inf\{f(X), X \in \Omega\} > 2.$$

Since for x fixed, $\lim_{y\to\infty} h(x,y) = h(x)$, there exists $M>0$ such that for all $x \geq \mu$, all y st $x^{-2}+y^{-2} < \frac{1}{4}$, $h(x,y) \leq h(x) + M\, h(y)$.

By the upper bound, for $\varepsilon>0$, there exists $n\geq 0$ st

$$\sum_{\Omega\setminus\Omega_n(r)} h(f(X)) \leq \frac{\varepsilon}{M}.$$

For $\vec{e} \in \mathcal{E}_{n+1}(r)$, we check that $F(\vec{e}) = h(x,y) \leq h(x) + M\, h(y)$ and hence by sum

$$1 = \sum_{\vec{e}\in \mathcal{E}_{n+1}(r)} F(\vec{e}) \leq 2 \sum_{X\in\Omega_n(r)} h(f(X)) + 2M \underbrace{\sum_{X\in\Omega_{n+1}(r)\setminus\Omega_n(r)} h(f(X))}_{\leq \varepsilon/M}$$

so $\displaystyle\sum_{X\in\Omega} h(f(X)) \geq \sum_{X\in\Omega_n(r)} h(f(X)) \geq \frac{1-\varepsilon}{2}$ which leads to the conclusion letting $\varepsilon\to 0$.