Geometry and spectrum of random hyperbolic surfaces

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Introduction

Aim Prove **properties** true for **typical** surfaces.

Motivation Large random regular graphs.

What properties?

- geometry:
  - diameter, injectivity radius, Cheeger constant [Mirzakhani 13]
  - length spectrum [Mirz.-Petri 17]
  - BS convergence, tangle-freeness [M. 20, M.-Thomas 20]

$\updownarrow$ Selberg trace formula

- spectrum of the Laplacian [M. 20, Gil-LeMa-Sahl-Tho 19]

"Typical"?

several possible point of views:

- generic surfaces
- discrete probabilistic construction [Brooks-Makover 04]
- random covers [Magee-Naud-Puder 20]
- probability density on set of surfaces $\rightsquigarrow$ our approach [Mirzakhani 07, 13]

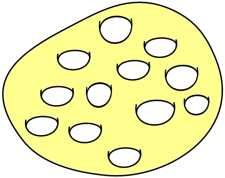
Probability on the set of surfaces

The moduli space

Idea: equip the **moduli space**

$$\mathcal{M}_g = {}^{\{\text{hyperbolic surfaces of genus } g\}}\!\big/_{\text{isometries}}.$$

with a **probability measure** $\mathbb{P}_g^{\mathrm{WP}}$.



High-genus limit

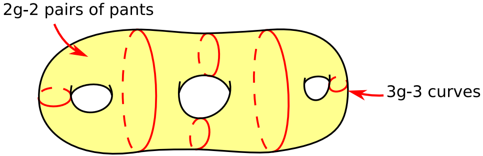
The volume of a hyperbolic surface of genus g is $2\pi(2g-2)$.
$\rightsquigarrow$ take the genus g going to infinity.
We will prove events true **with high probability** (w.h.p), i.e. such that

$$\lim_{g\to+\infty} \mathbb{P}_g^{\mathrm{WP}}(\mathit{event}) = 1.$$

Parametrisation the set of surfaces

Fenchel-Nielsen coordinates

Fix a topological surface of genus g, with a **pair of pant** decomposition.



Parametrisation the set of surfaces

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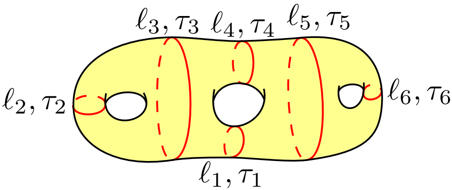
Fix a topological surface of genus g, with a **pair of pant** decomposition. Define a hyperbolic surface by picking a **length** ℓ_i and a **twist angle** τ_i for each curve of the pant decomposition.



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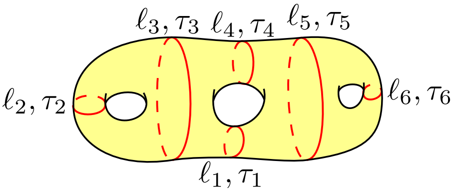

$\boxed{\text{Any hyperbolic surface is isometric to a surface obtained this way.}}$

$\rightsquigarrow$ The **Teichmüller space** $\mathscr{T}_g = \underbrace{(\mathbb{R}_+^*)^{3g-3}}_{\text{lengths}} \times \underbrace{\mathbb{R}^{3g-3}}_{\text{twists}}$ is the universal cover of the moduli space $\mathscr{M}_g = {}^{\mathscr{T}_g}\!/_{\text{isometry}}$.

Weil-Petersson probability

Wolpert's magic theorem

The natural symplectic form

$$\omega_g^{\mathrm{WP}} = \sum_{i=1}^{3g-3} \mathrm{d}\ell_i \wedge \mathrm{d}\tau_i$$

on $\mathscr{T}_g = (\mathbb{R}_{>0})^{3g-3} \times \mathbb{R}^{3g-3}$ is **invariant by the action of isometries**.
$\rightsquigarrow$ Natural symplectic structure on $\mathscr{M}_g$, with nice geometric properties.

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Probability measure

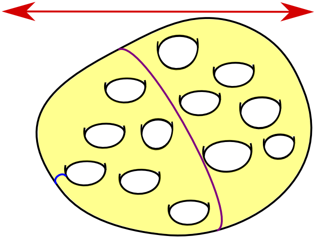
Symplectic form $\rightsquigarrow$ **volume form** $\mathrm{Vol}_g^{\mathrm{WP}} = \frac{1}{(3g-3)!}(\omega_g^{\mathrm{WP}})^{\wedge(3g-3)}$.

The total volume of $\mathscr{M}_g$ for $\mathrm{Vol}_g^{\mathrm{WP}}$ is **finite**.
We can **normalise** and obtain a probability $\mathbb{P}_g^{\mathrm{WP}} = \frac{1}{\mathrm{Vol}_g^{\mathrm{WP}}(\mathscr{M}_g)} \mathrm{Vol}_g^{\mathrm{WP}}$.

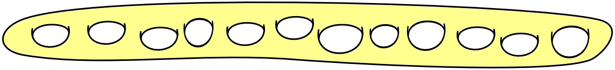
Geometry of typical surfaces of high genus

- **Diameter** [Mirzakhani 13]

$$\log(4g-2) \underset{\text{always}}{\leq} \mathrm{diam}(X) \underset{\text{w.h.p.}}{\leq} 40 \log g.$$



Atypical hyperbolic surface of high genus:



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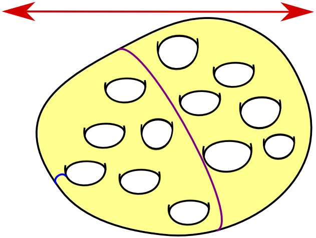

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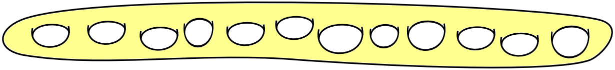

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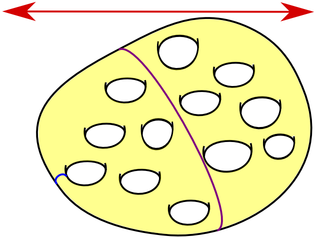

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- The number of primitive **closed geodesics** in $[a, b]$ is asymptotically a **Poisson law** of parameter $\int_a^b \frac{e^t + e^{-t} - 2}{t} \, dt$ [Mirzakhani Petri 17].

$$\mathbb{P}_g^{\mathrm{WP}}(\mathrm{injrad}(X) \leq \epsilon) \simeq \frac{\epsilon^2}{2} \qquad \text{for } \epsilon \ll 1 \text{ and } g \to +\infty.$$

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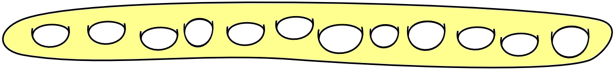
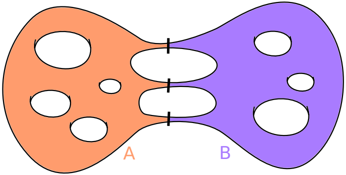


Cheeger constant

Definition

$$h = \inf_{A \sqcup B = X} \left\{ \frac{\text{length } \partial A}{\min(\text{area}(A), \text{area}(B))} \right\}$$

h large $\Leftrightarrow$ it is difficult to cut X.



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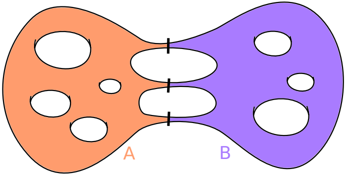

Typical hyperbolic surface

$\exists c > 0$ such that $h > c$ with high probability.

A first spectral consequence

Cheeger-Buser inequality: $\frac{h^2}{4} \leq \lambda_1 \leq h(1 + 10h)$.

$\rightsquigarrow$ uniform spectral gap for random hyperbolic surfaces!

Logarithmic-scale information

Motivation

- In spectral theory, we need to understand geodesics of length $\sim \log g$.
- Idea: we could assume that

$$\mathrm{injrad}(X) \geq \log g.$$

⚠ It is unlikely in this probabilistic setting...

- Solution: find ways to say that these small geodesics are **rare**.

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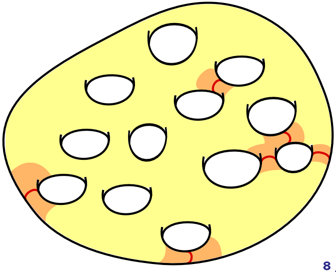
Theorem (Mirzakhani 13, M. 20)

W.h.p., random surfaces converge in the sense of ***Benjamini-Schramm*** *to the hyperbolic plane:*

$$\frac{\mathrm{Vol}\left(\left\{z \in X \,:\, \mathrm{injrad}_X(z) < \frac{1}{6}\log g\right\}\right)}{\mathrm{Vol}(X)} = O\left(g^{-\frac{1}{3}}\right).$$



The tangle-free hypothesis

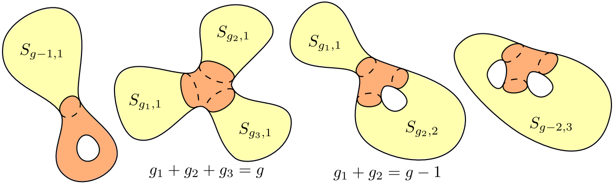
Definition (M.-Thomas 20)

We say a surface is *L-tangle-free* if it has no embedded pair of pant or handle of total boundary length $< 2L$.

Theorem (M.-Thomas 20)

- $\forall a < 1$, *with high probability, random surfaces are* $(a \log g)$*-TF.*

Proof: Use Mirzakhani's integration formula to compute the probability of each topological possibility:



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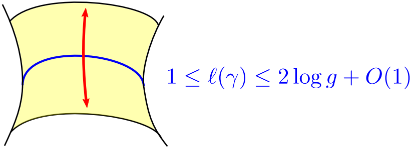
- *Any surface is $(4 \log g + O(1))$-tangled.*

Proof: Expand a neighbourhood around a 'short' closed geodesic γ.



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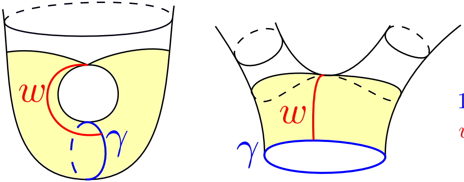
Proof: Expand a neighbourhood around a 'short' closed geodesic γ.
For some width w, the topology changes $\rightsquigarrow$ embedded pair of pant or handle.

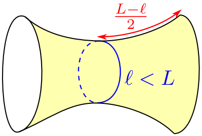


$1 \leq \ell(\gamma) \leq 2 \log g + O(1)$
$w \leq \log g + O(1)$

Theorem (M.-Thomas 20)

Let X be a L-tangle-free hyperbolic surface.

- ***Improved collar lemma***: *the $\frac{L-\ell}{2}$-neighbourhood of a closed geodesic of length $\ell < L$ is isometric to a cylinder.*



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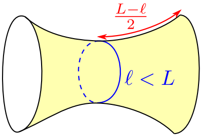
- *All closed geodesics of length $< \frac{L}{2}$ are pairwise disjoint.*



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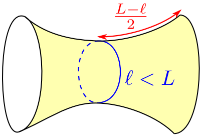
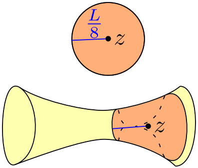
- *All closed geodesics of length $< L$ are simple.*
- *All closed geodesics of length $< \frac{L}{2}$ are pairwise disjoint.*
- *For all $z \in X$, the ball of center z and radius $\frac{L}{8}$ is isometric to:*
  - *a ball in the hyperbolic plane if $\mathrm{injrad}(z) > \frac{L}{4}$*
  - *a ball in a cylinder otherwise.*





Spectral theory of compact hyperbolic surfaces

Laplacian

X Riemannian manifold $\rightsquigarrow$ **Laplace-Beltrami operator** Δ on $L^2(X)$.
On the hyperbolic plane,

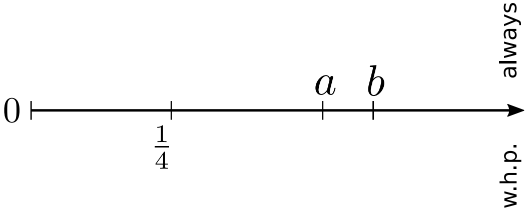
$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Spectral theorem

If X is **compact**, then there is an orthonormal basis of $L^2(X)$ of smooth eigenfunctions ϕ_j, such that $\Delta \phi_j = \lambda_j \phi_j$ and

$$0 = \lambda_0 < \lambda_1 \leq \ldots \leq \lambda_n \xrightarrow[n \to +\infty]{} +\infty.$$

Number of eigenvalues in a window

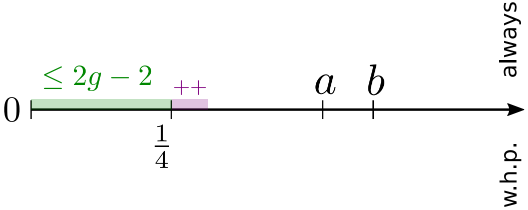


- 0 is always a simple eigenvalue, associated to constant functions.
- The spectrum of Δ on the hyperbolic plane is $\left[\frac{1}{4}, +\infty\right)$.
- Eigenvalues below $\frac{1}{4}$ are called **small eigenvalues**.
- For $0 \le a \le b$, we study the **counting function**

$$\mathcal{N}_X(a,b) = \#\{j : \lambda_j(X) \in [a,b]\}$$

We will compare what we can say for **all** surfaces and **most**.

Number of eigenvalues in a window



- $\mathcal{N}_X\left(0, \frac{1}{4}\right) \leq 2g - 2$. [Otal-Rosas 09]
- $\forall \epsilon > 0$, $\exists$ a surface X: $\mathcal{N}_X(0, \epsilon) = 2g - 2$. [Randol, Buser 70s]
- $\forall \epsilon > 0$, $\mathcal{N}_X\left(0, \frac{1}{4} + \epsilon\right)$ cannot be bounded in terms of g only. [Buser 77]

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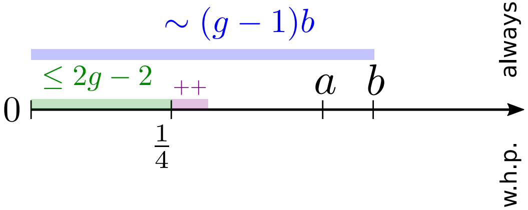

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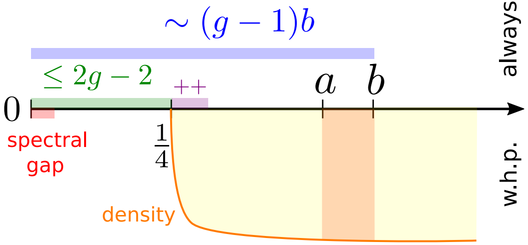
- $\forall \epsilon > 0$, $\mathcal{N}_X\left(0,\frac{1}{4}+\epsilon\right)$ cannot be bounded in terms of g only. [Buser 77]
- **Weyl law** [Bérard, Randol 78]

$$\frac{\mathcal{N}_X(0,b)}{\mathrm{Vol}(X)} = \frac{b}{4\pi} + O\left(\frac{\sqrt{b}}{\log b}\right) \qquad \text{as } b \to +\infty.$$

The implied constant depends on the surface.

Number of eigenvalues in a window



- **Uniform spectral gap** [Mirzakhani 13] (by Cheeger inequality)

$$\lambda_1 \geq 0.002467 \qquad \text{w.h.p.}$$

- Estimate on the **counting function** [M. 20]:

$$\frac{\mathcal{N}_X(a,b)}{\mathrm{Vol}(X)} \sim \mu(a,b) \qquad \text{as } g \text{ and/or } b \to +\infty.$$

for a measure μ supported on $[\frac{1}{4}, +\infty)$.
$\rightsquigarrow$ Similar to [McKay 81] for random regular graphs.

"Precise" statements

Theorem (M. 20)

$\exists$ a sequence of sets $\mathscr{A}_g \subset \mathscr{M}_g$ satisfying the following.

- $\mathbb{P}_g^{\mathrm{WP}}(\mathscr{A}_g) \underset{g \to +\infty}{\longrightarrow} 1.$
- *For any large enough g, any $0 \leq a \leq b$ and any $X \in \mathscr{A}_g$,*
  - *if $b \leq \frac{1}{4}$, then*

$$\frac{\mathcal{N}_X(0,b)}{\operatorname{Vol} X} = O\left(\frac{g^{-2^{-15}(\frac{1}{4}-b)^2}}{(\log g)^{\frac{3}{4}}}\right).$$

Corollary: *Then the number of small eigenvalues is $O\left(g(\log g)^{-\frac{3}{4}}\right)$.*

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Corollary: *Then the number of small eigenvalues is $O\left(g(\log g)^{-\frac{3}{4}}\right)$.*

  - *as soon as $a \geq \frac{1}{4}$ and $(b-a)\sqrt{\frac{\log g}{b+1}} \to +\infty$,*

$$\frac{\mathcal{N}_X(a,b)}{\mathrm{Vol}(X)} \sim \frac{1}{4\pi}\int_{\frac{1}{4}}^{+\infty} \mathbb{1}_{[a,b]}(\lambda) \tanh\left(\pi\sqrt{\lambda - \frac{1}{4}}\right) \mathrm{d}\lambda.$$

Corollary: *Uniform Weyl law + multiplicity estimate.*

Idea of the proof

Method from [Le Masson-Sahlsten 17].

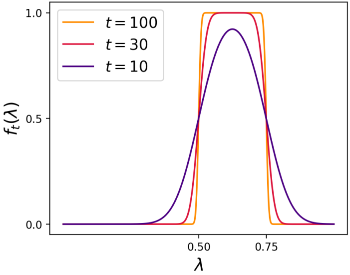
Selberg pretrace formula

$$\frac{1}{\mathrm{Vol}(X)} \sum_j f(\lambda_j) = \frac{1}{4\pi} \int_{\frac{1}{4}}^{+\infty} f(\lambda) \tanh\left(\pi \sqrt{\lambda - \frac{1}{4}}\right) \mathrm{d}\lambda + R(X, f).$$

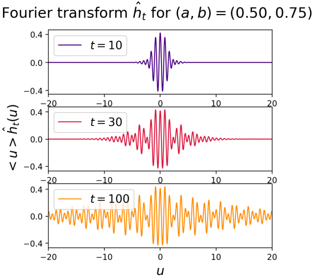
Test function

Smooth functions f_t converging to the step function $\mathbb{1}_{[a,b]}$ as $t \to +\infty$.

Test function f_t for $(a, b) = (0.50, 0.75)$



Idea of the proof: remainder estimate



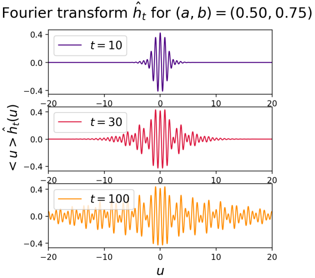
For $X = \mathscr{H} / \Gamma$, the remainder is:

$$R(X, f) = \int_D \sum_{\gamma \in \Gamma \setminus \{\mathrm{id}\}} K(z, \gamma \cdot z) \mathrm{d}z$$

where:

- K depends on the **Fourier transform** of the test function.
  - $\rightsquigarrow$ exponential decay
  - $\rightsquigarrow$ spreading as t increases.

Idea of the proof: remainder estimate



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where:

- K depends on the **Fourier transform** of the test function.
  - $\rightsquigarrow$ exponential decay
  - $\rightsquigarrow$ spreading as t increases.
- D is a fundamental domain.
  - $\mathrm{Good}(D)$ = points of **large injectivity radius** $\rightsquigarrow$ small contribution.
  - by Benjamini-Schramm convergence, $\mathrm{Vol}(\mathrm{Bad}(D))$ small.

Perpectives

Conjecture (Wright 19)

$\forall \epsilon > 0$,

$$\mathbb{P}_g^{\mathrm{WP}}\left(X \textit{ has an eigenvalue in } \left]0, \frac{1}{4} - \epsilon\right[\right) \underset{g \to +\infty}{\longrightarrow} 0.$$

- Equivalent of the Alon-Friedman theorem for large regular graphs. [Friedman 08, Bordenave 15]
- Joint work with Nalini Anantharaman.

Difficulties

- **Largest scale** on the surface:

$$\frac{1}{6} \log g \quad \longrightarrow \quad A \log g, \, A \text{ arbitrarily large.}$$

- Use the **oscillations** of the kernel and not only the exponential decay.

Thank you for your attention!